Towards an Expanded Definition of Consonance: Tuneable Intervals on Horn, Tuba and Trombone
by Marc Sabat and Robin Hayward

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# Towards an Expanded Definition of Consonance: Tuneable Intervals on Horn, Tuba and Trombone 

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#### Abstract

Three classes of musical intervals may be distinguished: tuneable intervals which may be tuned precisely by ear; intervals which cannot be directly tuned but may be reached through a finite series of tuneable interval steps; and intervals which cannot be precisely tuned, even if they can be closely approximated by a trained listener. The process of tuning precisely is accomplished by listening to the periodicity of the composite sound and by paying attention to combination tones and beats. Tuneable intervals are always expressible as frequency ratios of natural numbers.

This distinction offers the possibility of new, phenomenologically based definitions: a tuneable interval is called a consonance; an interval which cannot be precisely tuned is called a dissonance. The boundary between these classes will vary according to individual experience. Since some intervals are more readily tuned than others, relative consonance is proportional to the degree of difficulty in precisely tuning an interval.

The valved brass instruments derive their pitches from the overtone series of different lengths of tubing. By tuning these lengths in rational proportions to each other, the overtone series become related by frequency ratios of natural numbers, implying the possibility of producing tuneable intervals without having to alter the natural pitches. This also increases the potential number of tuneable intervals between two such instruments. Various options for horn, tuba and trombone are discussed in order to determine the most useful microtonal gamut, including conventional valve configurations as well as possible modifications of the instruments' design.

The set of available tuneable intervals within this large range of pitches is most readily determined by computational methods. The resulting tables offer an overview of new instrumental possibilities and establish a framework for composition within a 23 -limit subset of harmonic space. The ensuing discussion includes some general aspects of modulation and enharmonic near-equivalence in this space.


## 1. TUNEABLE SOUNDS

### 1.1 Definitions

When a sound exhibits a periodic or near-periodic pattern of vibrations, its spectral structure resembles a harmonic series, consisting of integer multiples of a fundamental frequency. A sound of this form may be called harmonic. Harmonic sounds which are perceived as having a single predominant pitch-height (the fundamental frequency) will be referred to as pitches. This paper describes research about the perceptibility of relationships between pitches. Computational methods allow musically significant subsets to be determined and compared with available microtonal pitch-sets on acoustic brass instruments.

The relationship between two pitches is called a musical interval. There is a perceptual identity between intervals having the same frequency ratio because of the analogy between their composite vibration patterns and their composite spectral structures. If a frequency ratio is represented by a ratio of natural numbers, the composite sound is also harmonic and is called a natural interval. The melodic distance between the pitches of an interval is commonly measured in cents proportional to the logarithm of the frequency ratio.

### 1.2 Ordering intervals melodically and harmonically

These two numerical representations (as a ratio of frequencies and as a distance in cents) suggest two ways in which natural intervals may be ordered. Let us assume two pitches $\mathrm{P}_{0}$ and $P_{1}$, with frequencies $F_{0}$ and $F_{1}$. If $P_{0}$ is allowed to remain fixed (for example as A4 440 Hz ) and $P_{1}$ varies within the range of the piano, from A0 ${ }^{1} 27.5 \mathrm{~Hz}$ up to C 84186 Hz , then the melodic distance

$$
\begin{equation*}
\phi\left(\mathrm{P}_{1}\right)=\left(1200 \cdot \log _{2}\left(\mathrm{~F}_{1} / \mathrm{F}_{0}\right)\right) \tag{1}
\end{equation*}
$$

varies between $-4800 \phi$ and $+3900 \phi$. As this value increases, the pitch rises. If it is negative, $P_{1}$ is below $P_{0}$; if positive, $P_{1}$ is above $P_{0}$. Given any set of pitches $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$, their melodic distances to $\mathrm{P}_{0}$ allow us to deduce a rising order $\left\{\mathrm{P}_{1}, \mathrm{P}_{2}, \ldots, \mathrm{P}_{n}\right\}$ with melodic distances to $\mathrm{P}_{0}$ $\left\{\phi\left(\mathrm{P}_{1}\right), \phi\left(\mathrm{P}_{2}\right), \ldots, \phi\left(\mathrm{P}_{\mathrm{n}}\right)\right\}$ such that $\left.\phi\left(\mathrm{P}_{\mathrm{b}}\right) \geq \phi\left(\mathrm{P}_{\mathrm{a}}\right)\right)$ if ( $\mathrm{b}>\mathrm{a}$ ). Note also that since (by logarithmic identities)

$$
\begin{equation*}
\phi\left(\mathrm{F}_{\mathrm{b}} / \mathrm{F}_{\mathrm{a}}\right)=\phi\left(\left(\mathrm{F}_{\mathrm{b}} / \mathrm{F}_{0}\right)\left(\mathrm{F}_{0} / \mathrm{F}_{\mathrm{a}}\right)\right)=\phi\left(\mathrm{F}_{\mathrm{b}} / \mathrm{F}_{0}\right)-\phi\left(\mathrm{F}_{\mathrm{a}} / \mathrm{F}_{0}\right) \tag{2}
\end{equation*}
$$

the set of melodic distances also allows us to determine all intervals between pitches in the set. In general, given two pitch sets $\left\{P_{n}\right\}$ and $\left\{Q_{m}\right\}$ it is possible to form an array $\operatorname{MD}\{P \times Q\}$ such that the elements of $P$ and $Q$ are in rising order and $M D_{i, j}=\phi\left(F_{p i} / F_{Q j}\right)$. This array evaluates all possible intervals between the pitch sets as melodic distances.

A second possible ordering may be deduced from the representation by frequency ratios. Just as cents enable us to represent the percept melodic distance between pitches, the ratios will enable us to deduce harmonic distance, a mathematical measure of relative consonance (more precisely defined below).

In general, frequency ratios made up of smaller numbers (in lowest terms), such as 1/1, 2/1,

[^0]$3 / 1,3 / 2,4 / 3$, represent more easily perceived intervals, and frequency ratios made up of larger numbers (in lowest terms), such as $19 / 16$ or $32 / 27$, represent intervals which are more difficult to perceive. One attempt to generalize this mathematically is Leonhard Euler's gradus suavitatis (consonance grade) function ${ }^{2}$ (1739). Given a ratio R in lowest terms, expressed as a product of primes in the form:
\[

$$
\begin{equation*}
\mathrm{R}=p_{1}{ }^{k 1} \cdot p_{2}^{k 2} \cdot \ldots \cdot p_{n}^{k n} \tag{3}
\end{equation*}
$$

\]

Euler's function $G(R)$ is defined as:

$$
\begin{equation*}
\mathrm{G}(\mathrm{R})=1+\sum\left|k_{i}\right| \cdot\left(p_{i}-1\right) \tag{4}
\end{equation*}
$$

Ratios which produce lower consonance grade values are usually easier to tune than ratios with higher values.

### 1.3 Tuneable Intervals

Consider two pitches, melodically close to each other, sounding simultaneously. As they approach unison, we perceive amplitude modulation or beating which gradually slows down. At a certain point, this beating is replaced by a phenomenon of spectral fusion (perceptible periodicity), which may be likened to a focussing of the sound of the interval $1 / 1$. Any interval that may be determined by ear in a similar manner is called a tuneable interval. Note that one of the characteristics of tuneability is a region of tolerance in which the interval is perceived as being nearly in tune. The existence of such regions, together with the fact that the frequency range of human hearing is limited, implies that there can only be a finite number of tuneable intervals.

Tuneability is not proposed as an absolute property. It varies depending on the register, relative volume and timbre of the sounds, as well as on the experience of the listener. Nevertheless, it suggests a precise perceptual definition of consonance: namely, a consonant interval is one which may be precisely tuned by ear. Relative consonance may be described as the degree of difficulty in achieving a precise intonation.

It should be noted that we distinguish tuneable intervals from intervals which may played by memorizing their sound and approximating their size. There are many ratios which familiar usage allows us to play with acceptable accuracy, but it is not possible to say from their sound alone when they are exactly centered. Several examples which come to mind include the semitone 16/15, the wholetone $9 / 8$, and the minor seventh 16/9. Similarly, many instrumentalists can learn to reproduce the pitches of Equal Temperament, for example when playing with a piano. By contrast, the septimal wholetone $8 / 7$ and the minor sevenths 7/4 and 9/5 are directly tuneable.

### 1.4 Prime limits and harmonic space

Let us consider the set of positive rational numbers in lowest terms. Each such number may be considered to represent a unique natural interval and (as above) may be represented as a product of the form

$$
\begin{equation*}
\mathrm{R}=p_{1}{ }^{k 1} \cdot p_{2}^{k 2} \cdot \ldots \cdot p_{n}^{k n} \tag{3}
\end{equation*}
$$

[^1]where the $p_{i}$ are unique and increasing prime numbers. Following the musical terminology established by Harry Partch ${ }^{3}$, we will define $p_{n}$ as the limit of the interval. For example, the interval $5 / 3$ will be called a " 5 -limit interval", all natural intervals generated by the primes 2,3 , and 5 will be called the " 5 -limit set", and so on. The set of all $p_{n}$-limit intervals may be modelled using James Tenney's concept of harmonic space ${ }^{4}$ as an $n$-dimensional lattice with the primes $p_{1}, p_{2}, \ldots, p_{n}$ as axes. The interval R is located by the coordinates $\left(k_{1}, k_{2}, \ldots, k_{n}\right)$. A progression from one interval to another can be modeled as a movement in harmonic space. Tenney proposes a refinement of Euler's gradus suavitatis with his harmonic distance function:
\[

$$
\begin{equation*}
\mathrm{HD}(\mathrm{R})=\mathrm{C} \cdot \sum\left|k_{i}\right| \log _{2}\left(p_{i}\right) \tag{5}
\end{equation*}
$$

\]

where C is a constant.
Since the number of possible tuneable intervals is finite, they must be contained within some prime limit. The subset of harmonic space which may be traversed by tuneable interval steps will be called the perceptible harmonic space. We will attempt to establish a provisional set of tuneable intervals and examine some characteristics of the resulting space.

To determine the prime limit of tuneable intervals, it is necessary to examine all intervals of the form $p / a$, where $p$ is a prime and a is an integer less than $p$. If a tuneable interval exists for each prime less than $p$, and if $p$ is also found to be tuneable, all intervals within the limit $p$ may be reached by a finite number of tuneable interval steps.

As an example, for the prime 23 the tuneability of each interval $23 / 1,23 / 2$, up to $23 / 22$ is examined. Note that the interval $23 / 1$ is itself between four and five octaves ${ }^{5}$ in size (circa $5428.3 \phi)$. Because the phenomena which establish tuneability depend largely on the interaction of spectral components, wider intervals are increasingly difficult to tune precisely. It is easier to judge the tuneability of narrower intervals, such as $23 / 3$ or $23 / 4$. Earlier research on string instruments ${ }^{6}$ limited the intervals for the purpose of musical practicability to a three-octave range. Nonetheless, a brief test of larger intervals indicates that ratios as high as $23 / 1$ could be tuneable and examination of these larger intervals merits further study. ${ }^{7}$

### 1.5 Analysis of the 23-limit set of tuneable intervals within three octaves

Let us assume a given prime $p$ is tuneable. In the set of intervals $\{p / a\}$ mentioned above, as the denominator rises the interval becomes narrower and the resulting periodic pattern increases in complexity. It is musically relevant to consider the smallest tuneable interval resulting from this series. Empiric testing indicates that for the seven primes $\{2,3,5,7,11,13,23\}$, at least one tuneable interval is $1200 \phi(2 / 1)$ or less. In the case of 17 and 19, the smallest tuneable interval is between $1200 \phi$ and $1902 \phi(3 / 1)$. For each prime $p$ up to 23, we can also define a least generative tuneable interval. It is the smallest p-limit tuneable interval with a power of 2 in the denominator: $\{2 / 1,3 / 2,5 / 4,7 / 4,11 / 8,13 / 8,17 / 4,19 / 8,23 / 8\}$.

[^2]The existence of these intervals implies that we can determine a finite tuneable path connecting any two points within a 23 -limit harmonic space.

Having established that we are primarily interested in intervals composed of the first nine prime numbers, the next step is to generate a set of possibly tuneable ratios and to subject them all to empirical testing. Intervals produced by the integers $\{1,2, \ldots, 28\}$ are considered. (Although it is possible that a few tuneable intervals may exist with ratios involving higher numbers, experience indicates that this is quite unlikely.) This set generates 784 ratios, 28 of which of the form $\mathrm{n} / \mathrm{n}$. The remaining 756 ratios may be paired into two groups of 378 , depending on whether the interval is measured downwards or upwards, for example $2 / 1$ and $1 / 2$. By reducing the set to intervals measured upwards (numerator $\geq$ denominator) 406 possibilities remain. For each interval, its melodic distance from $1 / 1(0 \phi)$ is calculated and the set is sorted in rising order. Considering only ratios in lowest terms up to 8/1 (3600ф), there are 213 possible tuneable intervals.

Each of these intervals was tested in several ways. Initially, the sounds were generated using a sampled reed organ timbre on computer and the pitch of the upper note was detuned in units of $0.1 \phi$ to see if the exact interval could be determined by ear. A second test involved listening for tuneable intervals as double-stops above the open $A$ and $D$ strings of a violin (slowly rising glissando). Whenever an interval could be tuned by eliminating beating or by periodicity recognition, its cents value would be measured using an electronic tuner and compared with the rising table. A similar test was performed above the open D string of a cello.

The resulting tuneable intervals were tested once again several months later, at which point 92 intervals were identified (including the unison 1/1). The 183 intervals above and below A were notated and their tuneability was further verified between two string instruments (violin and contrabass for the lower intervals and violin and violin for the upper intervals).

In the table below, the results of these tests are tabulated. The relative difficulty of tuning the sounds is shown in four degrees - easy (3), somewhat more difficult (2), quite difficult (1), not possible (0). Euler's consonance grade values and Tenney's harmonic distance values are included for comparison with the results of this empiric evaluation.

## Table 1

Limit Ratio Inversion Cents Tuneable Degree $\quad$ Euler $\quad$ Tenney

| 1 | 1/1 |  | 0 | x | 3 | 1 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 28/27 | 27/28 | 62.960904 |  | 0 | 15 | 9.56224242 |
| 13 | 27/26 | 26/27 | 65.337341 |  | 0 | 20 | 9.45532722 |
| 13 | 26/25 | 25/26 | 67.900234 |  | 0 | 22 | 9.34429591 |
| 5 | 25/24 | 24/25 | 70.672427 |  | 0 | 14 | 9.22881869 |
| 23 | 24/23 | 23/24 | 73.680654 |  | 0 | 28 | 9.10852446 |
| 23 | 23/22 | 22/23 | 76.956405 |  | 0 | 34 | 8.98299357 |
| 11 | 22/21 | 21/22 | 80.537035 |  | 0 | 20 | 8.85174904 |
| 7 | 21/20 | 20/21 | 84.467193 |  | 0 | 15 | 8.71424552 |
| 19 | 20/19 | 19/20 | 88.800698 |  | 0 | 25 | 8.56985561 |
| 19 | 19/18 | 18/19 | 93.603014 |  | 0 | 24 | 8.41785251 |
| 17 | 18/17 | 17/18 | 98.954592 |  | 0 | 22 | 8.25738784 |
| 17 | 17/16 | 16/17 | 104.95541 |  | 0 | 21 | 8.08746284 |
| 5 | 16/15 | 15/16 | 111.73129 |  | 0 | 11 | 7.9068906 |
| 7 | 15/14 | 14/15 | 119.44281 |  | 0 | 14 | 7.71424552 |
| 13 | 14/13 | 13/14 | 128.29824 |  | 0 | 20 | 7.50779464 |
| 5 | 27/25 | 25/27 | 133.23757 |  | 0 | 15 | 9.39874369 |
| 13 | 13/12 | 12/13 | 138.57266 |  | 0 | 17 | 7.28540222 |
| 23 | 25/23 | 23/25 | 144.35308 |  | 0 | 31 | 9.16741815 |
| 11 | 12/11 | 11/12 | 150.63706 |  | 0 | 15 | 7.04439412 |
| 23 | 23/21 | 21/23 | 157.49344 |  | 0 | 31 | 8.91587938 |
| 11 | 11/10 | 10/11 | 165.00423 |  | 0 | 16 | 6.78135971 |
| 19 | 21/19 | 19/21 | 173.26789 |  | 0 | 27 | 8.64024494 |
| 5 | 10/9 | 9/10 | 182.40371 |  | 0 | 10 | 6.4918531 |
| 19 | 19/17 | 17/19 | 192.55761 |  | 0 | 35 | 8.33539035 |
| 7 | 28/25 | 25/28 | 196.19848 |  | 0 | 17 | 9.45121111 |
| 3 | 9/8 | 8/9 | 203.91 |  | 0 | 8 | 6.169925 |
| 23 | 26/23 | 23/26 | 212.25331 |  | 0 | 36 | 9.22400167 |
| 17 | 17/15 | 15/17 | 216.68669 |  | 0 | 23 | 7.99435344 |
| 11 | 25/22 | 22/25 | 221.30949 |  | 0 | 20 | 9.10328781 |
| 7 | 8/7 | 7/8 | 231.17409 | x | 2 | 10 | 5.80735492 |
| 23 | 23/20 | 20/23 | 241.96063 |  | 0 | 29 | 8.84549005 |
| 13 | 15/13 | 13/15 | 247.74105 |  | 0 | 19 | 7.60733031 |
| 19 | 22/19 | 19/22 | 253.80493 |  | 0 | 30 | 8.70735913 |
| 7 | 7/6 | 6/7 | 266.87091 | x | 3 | 10 | 5.39231742 |
| 23 | 27/23 | 23/27 | 277.59066 |  | 0 | 29 | 9.27844946 |
| 17 | 20/17 | 17/20 | 281.3583 |  | 0 | 23 | 8.40939094 |
| 13 | 13/11 | 11/13 | 289.20972 |  | 0 | 23 | 7.15987134 |
| 19 | 19/16 | 16/19 | 297.51302 |  | 0 | 23 | 8.24792751 |
| 7 | 25/21 | 21/25 | 301.84652 |  | 0 | 17 | 9.03617361 |
| 5 | 6/5 | 5/6 | 315.64129 | x | 3 | 8 | 4.9068906 |
| 23 | 23/19 | 19/23 | 330.76133 |  | 0 | 41 | 8.77148947 |
| 17 | 17/14 | 14/17 | 336.1295 |  | 0 | 24 | 7.89481776 |
| 23 | 28/23 | 23/28 | 340.55156 |  | 0 | 31 | 9.33091688 |
| 11 | 11/9 | 9/11 | 347.40794 | x | 1 | 15 | 6.62935662 |
| 11 | 27/22 | 22/27 | 354.54706 |  | 0 | 18 | 9.21431912 |
| 13 | 16/13 | 13/16 | 359.47234 |  | 0 | 17 | 7.70043972 |
| 17 | 21/17 | 17/21 | 365.8255 |  | 0 | 25 | 8.47978026 |
| 13 | 26/21 | 21/26 | 369.74675 |  | 0 | 22 | 9.09275714 |


| 5 | 5/4 | 4/5 | 386.31371 | x | 3 | 7 | 4.32192809 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 19 | 24/19 | 19/24 | 404.44198 |  | 0 | 24 | 8.83289001 |
| 19 | 19/15 | 15/19 | 409.2443 |  | 0 | 25 | 8.15481811 |
| 11 | 14/11 | 11/14 | 417.50796 |  | 0 | 18 | 7.26678654 |
| 23 | 23/18 | 18/23 | 424.36435 |  | 0 | 28 | 8.69348696 |
| 7 | 9/7 | 7/9 | 435.0841 | x | 3 | 11 | 5.97727992 |
| 17 | 22/17 | 17/22 | 446.36253 |  | 0 | 28 | 8.54689446 |
| 13 | 13/10 | 10/13 | 454.21395 | x | 2 | 18 | 7.02236781 |
| 17 | 17/13 | 13/17 | 464.42775 |  | 0 | 29 | 7.78790256 |
| 7 | 21/16 | 16/21 | 470.78091 |  | 0 | 13 | 8.39231742 |
| 19 | 25/19 | 19/25 | 475.11441 |  | 0 | 27 | 8.8917837 |
| 3 | 4/3 | 3/4 | 498.045 | $\mathbf{x}$ | 3 | 7 | 3.5849625 |
| 5 | 27/20 | 20/27 | 519.55129 |  | 0 | 13 | 9.0768156 |
| 23 | 23/17 | 17/23 | 523.31894 |  | 0 | 39 | 8.6110248 |
| 19 | 19/14 | 14/19 | 528.68711 |  | 0 | 26 | 8.05528244 |
| 11 | 15/11 | 11/15 | 536.95077 |  | 0 | 17 | 7.36632221 |
| 19 | 26/19 | 19/26 | 543.01465 |  | 0 | 32 | 8.94836723 |
| 11 | 11/8 | 8/11 | 551.31794 | x | 1 | 14 | 6.45943162 |
| 13 | 18/13 | 13/18 | 563.38234 |  | 0 | 18 | 7.87036472 |
| 5 | 25/18 | 18/25 | 568.71743 |  | 0 | 14 | 8.81378119 |
| 7 | 7/5 | 5/7 | 582.51219 | x | 3 | 11 | 5.12928302 |
| 17 | 24/17 | 17/24 | 596.99959 |  | 0 | 22 | 8.67242534 |
| 17 | 17/12 | 12/17 | 603.00041 |  | 0 | 21 | 7.67242534 |
| 19 | 27/19 | 19/27 | 608.35199 |  | 0 | 25 | 9.00281502 |
| 7 | 10/7 | 7/10 | 617.48781 | $\mathbf{x}$ | 2 | 12 | 6.12928302 |
| 23 | 23/16 | 16/23 | 628.27435 |  | 0 | 27 | 8.52356196 |
| 13 | 13/9 | 9/13 | 636.61766 | x | 1 | 17 | 6.87036472 |
| 11 | 16/11 | 11/16 | 648.68206 | x | 1 | 15 | 7.45943162 |
| 19 | 19/13 | 13/19 | 656.98535 |  | 0 | 31 | 7.94836723 |
| 11 | 22/15 | 15/22 | 663.04923 |  | 0 | 18 | 8.36632221 |
| 17 | 25/17 | 17/25 | 667.67202 |  | 0 | 25 | 8.73131903 |
| 19 | 28/19 | 19/28 | 671.31289 |  | 0 | 27 | 9.05528244 |
| 3 | 3/2 | 2/3 | 701.955 | x | 3 | 4 | 2.5849625 |
| 17 | 26/17 | 17/26 | 735.57225 |  | 0 | 30 | 8.78790256 |
| 23 | 23/15 | 15/23 | 740.00563 |  | 0 | 29 | 8.43045255 |
| 13 | 20/13 | 13/20 | 745.78605 |  | 0 | 19 | 8.02236781 |
| 17 | 17/11 | 11/17 | 753.63747 |  | 0 | 27 | 7.54689446 |
| 7 | 14/9 | 9/14 | 764.9159 | $\mathbf{x}$ | 1 | 12 | 6.97727992 |
| 5 | 25/16 | 16/25 | 772.62743 |  | 0 | 13 | 8.64385619 |
| 11 | 11/7 | 7/11 | 782.49204 | $\mathbf{x}$ | 2 | 17 | 6.26678654 |
| 19 | 19/12 | 12/19 | 795.55802 |  | 0 | 23 | 7.83289001 |
| 17 | 27/17 | 17/27 | 800.90959 |  | 0 | 23 | 8.84235034 |
| 5 | 8/5 | 5/8 | 813.68629 | x | 3 | 8 | 5.32192809 |
| 13 | 21/13 | 13/21 | 830.25325 |  | 0 | 21 | 8.09275714 |
| 13 | 13/8 | 8/13 | 840.52766 | x | 2 | 16 | 6.70043972 |
| 11 | 18/11 | 11/18 | 852.59206 |  | 0 | 16 | 7.62935662 |
| 23 | 23/14 | 14/23 | 859.44844 |  | 0 | 30 | 8.33091688 |
| 17 | 28/17 | 17/28 | 863.8705 |  | 0 | 25 | 8.89481776 |
| 5 | 5/3 | 3/5 | 884.35871 | x | 3 | 7 | 3.9068906 |
| 3 | 27/16 | 16/27 | 905.865 |  | 0 | 11 | 8.7548875 |
| 13 | 22/13 | 13/22 | 910.79028 |  | 0 | 24 | 8.15987134 |


| 17 | 17/10 | 10/17 | 918.6417 |  | 0 | 22 | 7.40939094 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 12/7 | 7/12 | 933.12909 | x | 2 | 11 | 6.39231742 |
| 19 | 19/11 | 11/19 | 946.19507 |  | 0 | 29 | 7.70735913 |
| 13 | 26/15 | 15/26 | 952.25895 |  | 0 | 20 | 8.60733031 |
| 7 | 7/4 | 4/7 | 968.82591 | x | 3 | 9 | 4.80735492 |
| 23 | 23/13 | 13/23 | 987.74669 |  | 0 | 35 | 8.22400167 |
| 3 | 16/9 | 9/16 | 996.09 |  | 0 | 9 | 7.169925 |
| 7 | 25/14 | 14/25 | 1003.8015 |  | 0 | 16 | 8.45121111 |
| 5 | 9/5 | 5/9 | 1017.5963 | x | 3 | 9 | 5.4918531 |
| 11 | 20/11 | 11/20 | 1034.9958 |  | 0 | 17 | 7.78135971 |
| 11 | 11/6 | 6/11 | 1049.3629 | x | 2 | 14 | 6.04439412 |
| 13 | 24/13 | 13/24 | 1061.4273 |  | 0 | 18 | 8.28540222 |
| 13 | 13/7 | 7/13 | 1071.7018 | $\mathbf{x}$ | 2 | 19 | 6.50779464 |
| 7 | 28/15 | 15/28 | 1080.5572 |  | 0 | 15 | 8.71424552 |
| 5 | 15/8 | 8/15 | 1088.2687 | x | 1 | 10 | 6.9068906 |
| 17 | 17/9 | 9/17 | 1101.0454 |  | 0 | 21 | 7.25738784 |
| 19 | 19/10 | 10/19 | 1111.1993 |  | 0 | 24 | 7.56985561 |
| 11 | 21/11 | 11/21 | 1119.463 |  | 0 | 19 | 7.85174904 |
| 23 | 23/12 | 12/23 | 1126.3193 | x | 1 | 27 | 8.10852446 |
| 13 | 25/13 | 13/25 | 1132.0998 |  | 0 | 21 | 8.34429591 |
| 7 | 27/14 | 14/27 | 1137.0391 |  | 0 | 14 | 8.56224242 |
| 2 | 2/1 | 1/2 | 1200 | x | 3 | 2 | 1 |
| 13 | 27/13 | 13/27 | 1265.3373 |  | 0 | 19 | 8.45532722 |
| 5 | 25/12 | 12/25 | 1270.6724 |  | 0 | 13 | 8.22881869 |
| 23 | 23/11 | 11/23 | 1276.9564 |  | 0 | 33 | 7.98299357 |
| 7 | 21/10 | 10/21 | 1284.4672 |  | 0 | 14 | 7.71424552 |
| 19 | 19/9 | 9/19 | 1293.603 |  | 0 | 23 | 7.41785251 |
| 17 | 17/8 | 8/17 | 1304.9554 |  | 0 | 20 | 7.08746284 |
| 7 | 15/7 | 7/15 | 1319.4428 |  | 0 | 13 | 6.71424552 |
| 13 | 28/13 | 13/28 | 1328.2982 |  | 0 | 21 | 8.50779464 |
| 13 | 13/6 | 6/13 | 1338.5727 | x | 2 | 16 | 6.28540222 |
| 11 | 24/11 | 11/24 | 1350.6371 |  | 0 | 16 | 8.04439412 |
| 11 | 11/5 | 5/11 | 1365.0042 | x | 2 | 15 | 5.78135971 |
| 5 | 20/9 | 9/20 | 1382.4037 |  | 0 | 11 | 7.4918531 |
| 3 | 9/4 | 4/9 | 1403.91 | x | 3 | 7 | 5.169925 |
| 11 | 25/11 | 11/25 | 1421.3095 |  | 0 | 19 | 8.10328781 |
| 7 | 16/7 | 7/16 | 1431.1741 |  | 0 | 11 | 6.80735492 |
| 23 | 23/10 | 10/23 | 1441.9606 |  | 0 | 28 | 7.84549005 |
| 7 | 7/3 | 3/7 | 1466.8709 | x | 3 | 9 | 4.39231742 |
| 13 | 26/11 | 11/26 | 1489.2097 |  | 0 | 24 | 8.15987134 |
| 19 | 19/8 | 8/19 | 1497.513 | x | 1 | 22 | 7.24792751 |
| 5 | 12/5 | 5/12 | 1515.6413 | x | 3 | 9 | 5.9068906 |
| 17 | 17/7 | 7/17 | 1536.1295 | x | 1 | 23 | 6.89481776 |
| 11 | 22/9 | 9/22 | 1547.4079 |  | 0 | 16 | 7.62935662 |
| 11 | 27/11 | 11/27 | 1554.5471 |  | 0 | 17 | 8.21431912 |
| 5 | 5/2 | 2/5 | 1586.3137 | x | 3 | 6 | 3.32192809 |
| 11 | 28/11 | 11/28 | 1617.508 |  | 0 | 19 | 8.26678654 |
| 23 | 23/9 | 9/23 | 1624.3643 |  | 0 | 27 | 7.69348696 |
| 7 | 18/7 | 7/18 | 1635.0841 | x | 2 | 12 | 6.97727992 |
| 13 | 13/5 | 5/13 | 1654.2139 | x | 2 | 17 | 6.02236781 |
| 7 | 21/8 | 8/21 | 1670.7809 |  | 0 | 12 | 7.39231742 |


| 3 | 8/3 | 3/8 | 1698.045 | $\mathbf{x}$ | 3 | 6 | 4.5849625 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 27/10 | 10/27 | 1719.5513 |  | 0 | 12 | 8.0768156 |
| 19 | 19/7 | 7/19 | 1728.6871 |  | 0 | 25 | 7.05528244 |
| 11 | 11/4 | 4/11 | 1751.3179 | $\mathbf{x}$ | 3 | 13 | 5.45943162 |
| 5 | 25/9 | 9/25 | 1768.7174 |  | 0 | 13 | 7.81378119 |
| 7 | 14/5 | 5/14 | 1782.5122 | x | 3 | 12 | 6.12928302 |
| 17 | 17/6 | 6/17 | 1803.0004 | x | 2 | 20 | 6.67242534 |
| 7 | 20/7 | 7/20 | 1817.4878 | x | 2 | 13 | 7.12928302 |
| 23 | 23/8 | 8/23 | 1828.2743 | x | 2 | 26 | 7.52356196 |
| 13 | 26/9 | 9/26 | 1836.6177 |  | 0 | 18 | 7.87036472 |
| 3 | 3/1 | 1/3 | 1901.955 | x | 3 | 3 | 1.5849625 |
| 7 | 28/9 | 9/28 | 1964.9159 | x | 1 | 13 | 7.97727992 |
| 5 | 25/8 | 8/25 | 1972.6274 | $\mathbf{x}$ | 1 | 12 | 7.64385619 |
| 11 | 22/7 | 7/22 | 1982.492 | x | 1 | 18 | 7.26678654 |
| 19 | 19/6 | 6/19 | 1995.558 | x | 1 | 22 | 6.83289001 |
| 5 | 16/5 | 5/16 | 2013.6863 | x | 3 | 9 | 6.32192809 |
| 13 | 13/4 | 4/13 | 2040.5277 | $\mathbf{x}$ | 3 | 15 | 5.70043972 |
| 23 | 23/7 | 7/23 | 2059.4484 |  | 0 | 29 | 7.33091688 |
| 5 | 10/3 | 3/10 | 2084.3587 | x | 3 | 9 | 5.4918531 |
| 3 | 27/8 | 8/27 | 2105.865 | $\mathbf{x}$ | 1 | 10 | 7.7548875 |
| 17 | 17/5 | 5/17 | 2118.6417 | $\mathbf{x}$ | 2 | 21 | 6.40939094 |
| 7 | 24/7 | 7/24 | 2133.1291 | x | 1 | 12 | 7.39231742 |
| 7 | 7/2 | 2/7 | 2168.8259 | $\mathbf{x}$ | 3 | 8 | 3.80735492 |
| 7 | 25/7 | 7/25 | 2203.8015 |  | 0 | 15 | 7.45121111 |
| 5 | 18/5 | 5/18 | 2217.5963 | x | 3 | 10 | 6.4918531 |
| 11 | 11/3 | 3/11 | 2249.3629 | $\mathbf{x}$ | 3 | 13 | 5.04439412 |
| 13 | 26/7 | 7/26 | 2271.7018 |  | 0 | 20 | 7.50779464 |
| 5 | 15/4 | 4/15 | 2288.2687 | $\mathbf{x}$ | 3 | 9 | 5.9068906 |
| 19 | 19/5 | 5/19 | 2311.1993 | x | 2 | 23 | 6.56985561 |
| 23 | 23/6 | 6/23 | 2326.3193 | x | 1 | 26 | 7.10852446 |
| 7 | 27/7 | 7/27 | 2337.0391 | $\mathbf{x}$ | 1 | 13 | 7.56224242 |
| 2 | 4/1 | 1/4 | 2400 | x | 3 | 5 | 2 |
| 5 | 25/6 | 6/25 | 2470.6724 | $\mathbf{x}$ | 1 | 12 | 7.22881869 |
| 7 | 21/5 | 5/21 | 2484.4672 | x | 1 | 13 | 6.71424552 |
| 17 | 17/4 | 4/17 | 2504.9554 | x | 2 | 19 | 6.08746284 |
| 13 | 13/3 | 3/13 | 2538.5727 | x | 3 | 15 | 5.28540222 |
| 11 | 22/5 | 5/22 | 2565.0042 | x | 1 | 16 | 6.78135971 |
| 3 | 9/2 | 2/9 | 2603.91 | $\mathbf{x}$ | 3 | 6 | 4.169925 |
| 23 | 23/5 | 5/23 | 2641.9606 | $\mathbf{x}$ | 2 | 27 | 6.84549005 |
| 7 | 14/3 | 3/14 | 2666.8709 | x | 3 | 10 | 5.39231742 |
| 19 | 19/4 | 4/19 | 2697.513 | x | 2 | 21 | 6.24792751 |
| 5 | 24/5 | 5/24 | 2715.6413 | x | 2 | 10 | 6.9068906 |
| 5 | 5/1 | 1/5 | 2786.3137 | x | 3 | 5 | 2.32192809 |
| 13 | 26/5 | 5/26 | 2854.2139 | x | 1 | 18 | 7.02236781 |
| 7 | 21/4 | 4/21 | 2870.7809 | $\mathbf{x}$ | 2 | 11 | 6.39231742 |
| 3 | 16/3 | 3/16 | 2898.045 | $\mathbf{x}$ | 3 | 7 | 5.5849625 |
| 5 | 27/5 | 5/27 | 2919.5513 |  | 0 | 11 | 7.0768156 |
| 11 | 11/2 | 2/11 | 2951.3179 | x | 3 | 12 | 4.45943162 |
| 7 | 28/5 | 5/28 | 2982.5122 | $\mathbf{x}$ | 2 | 13 | 7.12928302 |
| 17 | 17/3 | 3/17 | 3003.0004 | $\mathbf{x}$ | 3 | 19 | 5.67242534 |
| 23 | 23/4 | 4/23 | 3028.2743 | x | 2 | 25 | 6.52356196 |


| 3 | 6/1 | 1/6 | 3101.955 | x | 3 | 4 | 2.5849625 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 25/4 | 4/25 | 3172.6274 | x | 2 | 11 | 6.64385619 |
| 19 | 19/3 | 3/19 | 3195.558 | X | 3 | 21 | 5.83289001 |
| 13 | 13/2 | 2/13 | 3240.5277 | X | 3 | 14 | 4.70043972 |
| 5 | 20/3 | 3/20 | 3284.3587 | x | 3 | 9 | 5.9068906 |
| 3 | 27/4 | 4/27 | 3305.865 | x | 1 | 9 | 6.7548875 |
| 7 | 7/1 | 1/7 | 3368.8259 | X | 3 | 7 | 2.80735492 |
| 11 | 22/3 | 3/22 | 3449.3629 | x | 3 | 14 | 6.04439412 |
| 5 | 15/2 | 2/15 | 3488.2687 | X | 3 | 8 | 4.9068906 |
| 23 | 23/3 | 3/23 | 3526.3193 | x | 3 | 25 | 6.10852446 |
| 2 | 8/1 | 1/8 | 3600 | x | 3 | 4 | 3 |

The set of tuneable intervals above A4, written in musical staff notation, follows as Table 2. The accidentals, which provide exact graphical representations of frequency ratios, are from the Extended Helmholtz-Ellis JI Pitch Notation ${ }^{8}$. A legend of the accidentals is included as Table 3. The three degrees of tuneability are notated by using large open notes (3), small open notes (2) and small black notes (1) respectively.

These intervals represent a set of microtonal interval-classes which are congruent to the ability to perceive relationships between harmonic sounds and serve as a basis for further analysis. Note that although the subset above $1 / 1$ and the subset below $1 / 1$ mirror each other exactly, different classes of intervals are present in each octave (for example, 13/3, 3/13, $13 / 6$ and $6 / 13$ are included, but $13 / 12$ and $12 / 13$ are not). This is a departure from the classical convention of allowing pitch-class set representations in which elements may be freely transposed between octaves.

Tables 2 and 3 (overleaf)

[^3]
# INTERVALS TUNEABLE BY EAR (up to 3 Octaves wide) 

below the violin A-string
tested and notated by Marc Sabat
with assistance from Wolfgang von Schweinitz (cello); Beltane Ruiz (bass); Anaïs Chen (violin)





## The Extended Helmholtz-Ellis JI Pitch Notation

microtonal accidentals designed by Marc Sabat and Wolfgang von Schweinitz, 2004

## 3-LIMIT (PYTHAGOREAN) INTERVALS

 $b b \quad b \quad \ddagger \quad \# \quad x$
## FUNCTION OF THE ACCIDENTALS

notate 35 pitches from the series of untempered perfect fifths $(3 / 2) \approx \pm 702.0$ cents;
perfect fifth (3/2); perfect fourth (4/3); major wholetone (9/8)

## 5-LIMIT (PTOLEMAIC) INTERVALS


notate an alteration by one syntonic comma (81/80) $\approx \pm 21.5$ cents; major third (5/4); minor third (6/5); major sixth (5/3); minor sixth (8/5); minor wholetone (10/9)
notate an alteration by two syntonic commas
$(81 / 80) \cdot(81 / 80) \approx \pm 43.0$ cents;
augmented fifth (25/16); diminished fourth (32/25)
notate an alteration by three syntonic commas
$(81 / 80) \cdot(81 / 80) \cdot(81 / 80) \approx \pm 64.5$ cents;
minor diesis (128/125)

7-LIMIT (SEPTIMAL) INTERVALS

notate an alteration by one septimal comma (64/63) $\approx \pm 27.3$ cents; natural seventh (7/4); septimal wholetone (8/7); septimal diminished fifth (7/5); septimal tritone (10/7); septimal minor third (7/6); septimal quartertone (36/35)
notate an alteration by two septimal commas
(64/63) $\cdot(64 / 63) \approx \pm 54.5$ cents;
septimal sixthtone (49/48)

## 11-LIMIT (UNDECIMAL) INTERVALS


notate an alteration by one undecimal quartertone
$(33 / 32) \approx \pm 53.3$ cents;
undecimal augmented fourth (11/8); undecimal diminished fifth (16/11)

13-LIMIT (TRIDECIMAL) INTERVALS

notate an alteration by one tridecimal thirdtone $(27 / 26) \approx \pm 65.3$ cents; tridecimal neutral sixth (13/8); tridecimal neutral third (16/13)

PRIMES IN THE HARMONIC SERIES OCTAVE 16-32 (5-limit signs are given here relative to "A")
notate an alteration of the 5 -limit accidental by one 17 -limit schisma $(16 / 17) \cdot(16 / 15)=(256 / 255) \approx \pm 6.8$ cents;
Galileo's "equal-tempered" semitone (18/17);
17-limit diminshed seventh chord 10:12:14:17
notate an alteration by one 19-limit schisma
$(19 / 16) \cdot(27 / 32)=(513 / 512) \approx \pm 3.4$ cents;
19-limit minor third (19/16); 19-limit minor triad 16:19:24
notate an alteration by one 23 -limit comma
$(23 / 16) \cdot(8 / 9) \cdot(8 / 9) \cdot(8 / 9) \approx \pm 16.5$ cents;
raised leading tone (23/12)
$\uparrow \uparrow \quad \downarrow \emptyset$
notate an alteration of the 5 －limit accidental by one 29－limit comma $(29 / 16) \cdot(5 / 9)=(145 / 144) \approx \pm 12.0$ cents
notate an alteration of the 11－limit accidental by one 31－limit schisma $(32 / 31) \cdot(32 / 33)=(1024 / 1023) \approx \pm 1.7$ cents

PRIMES IN THE HARMONIC SERIES OCTAVE 32－64（5－limit signs are given here relative to＂$A$＂）

| $\{$ ミ\} $\downarrow$ | $\{=\} d$ | notate an alteration of the 11－limit accidental by one 37 －limit schisma $(36 / 37) \cdot(33 / 32)=(297 / 296) \approx \pm 5.8$ cents |
| :---: | :---: | :---: |
| $\{\underset{\#}{\psi}\}$ | $\{p\}$ | notate an alteration of the 5 －limit accidental by one 41－limit schisma $(32 / 41) \cdot(81 / 64) \cdot(81 / 80)=(6561 / 6560) \approx \pm 0.3$ cents |
| $\{\uparrow\}$ | $\{\downarrow\}$ | notate an alteration by one 43 －limit comma $(43 / 32) \cdot(3 / 4)=(129 / 128) \approx \pm 13.5$ cents |
| $\{\boldsymbol{N}\} \#$ or $\{\boldsymbol{\sim}$ | $\{b\} b$ or $\{b b\} b$ | notate an alteration of the 7 －limit accidental by one 47 －limit schisma $(32 / 47) \cdot(48 / 49) \cdot(3 / 2)=(2304 / 2303) \approx \pm 0.8$ cents |
| $\{ミ\} \#$ | $\{彡\} \hat{b}$ | notate an alteration of the 5 －limit accidental by one 53－limit comma $(32 / 53) \cdot(5 / 3)=(160 / 159) \approx \pm 10.9$ cents |
| $\{\\|\}$ | $\{d\}$ | notate an alteration of the 13 －limit accidental by one 59 －limit schisma $(32 / 59) \cdot(24 / 13)=(768 / 767) \approx \pm 2.3$ cents |
| $\{\boldsymbol{p}\}$ | $\{b\} \hat{b}$ | notate an alteration of the 7 －limit accidental by one 61 －limit schisma $(61 / 32) \cdot(21 / 40)=(1281 / 1280) \approx \pm 1.4$ cents |

## IRRATIONAL AND TEMPERED INTERVALS

bb $\quad \begin{array}{llll}\square & \square & \quad \\ \square\end{array}$
notate the respective Equal Tempered Semitone； may be combined with a cents indication to notate any pitch

## NOTE ABOUT CENTS INDICATIONS

optional cents indications may be placed above or below the respective accidentals and are always understood in reference to Equal Tempered semitones，as implied by the Pythagorean accidentals

### 1.6 Analytic applications of the tuneable intervals

Before moving to a discussion about the realization of tuneable intervals on acoustic brass instruments, we conclude this section by mentioning several properties of the intervals which merit further investigation.

It was noted above that given any two pitches within 23-limit harmonic space, it is possible to determine a finite tuneable path between them using least generative tuneable intervals of the form $\left(p / 2^{n}\right)$. This is easily demonstrated as follows. The two pitches are products of primes up to 23. Thus, the interval between them is also a product of primes up to 23. Each such prime may be written as a product of a generative tuneable interval and $2^{n} / 1$. These products may be combined into a new representation of the interval, in which factors of the form $2^{n} / 1$ are reduced to $(2 / 1)^{m}$. This result is the desired tuneable path.

Since we are moving by discrete steps in harmonic space, there must also be a shortest tuneable path. For example, all directly tuneable intervals have a shortest tuneable path of 1. The interval $9 / 8$ has a shortest tuneable path of 2 since $(9 / 8)=(9 / 4) \cdot(1 / 2)$. The set of 23 -limit intervals with a shortest tuneable path $\geq 2$ may be defined as the set of intervals indirectly tuneable by construction. Evaluating the shortest tuneable path and grouping intervals based on this value indicates one way of developing the concept of harmonic distance.

It would be conceivable to use this concept to develop a generalized model of tonality and harmonic regions in the sense Schoenberg proposes in his "Structural Functions of Harmony" ${ }^{\prime 9}$. Given an arbitrary 23 -limit pitch $\mathrm{P}_{0}$, consider the set of pitches within tuneable path 1 and call it the tuneable harmonic region $\left\{\operatorname{THR}\left(\mathrm{P}_{0}\right)\right\}$. The set of all chords generated by these pitches may be parsed into classes of similar structures (inverted, transposed) built on different degrees. The process of modulation might be well defined as a movement to a new tuneable harmonic region.

Another significant application of tuneable intervals is the principle of tuneable melodic steps. A melodic step is defined as being tuneable if it is the difference between two tuneable intervals. In musical terms, if a pitch is sustained in one voice while a second voice moves from one tuneable interval to another, both pitches may be precisely tuned by definition and thus the melodic step between them may be accurately tuned. By evaluating the set of all tuneable melodic steps, it has been determined that the largest tuneable melodic step is $8 / 1$ ( $3600 \phi$ ) and the smallest is $144 / 143$ (12.1 $\phi$ ). Each such step may have several realizations, which represent different musical possibilities. For example, the melodic interval $9 / 8$ can be tuned in 19 distinct ways as the difference between two tuneable intervals.

Finally, these intervals can serve as a basis for analyzing more complex tuneable structures consisting of more than two pitches. In the case of three voices, a triad ( $\left.P_{1}: P_{2}: P_{3}\right)$ always has the three component intervals $\mathrm{P}_{1}: \mathrm{P}_{2}, \mathrm{P}_{2}: \mathrm{P}_{3}$ and $\mathrm{P}_{1}: \mathrm{P}_{3}$. It is possible to define a set of triads in which each component interval is tuneable and to examine the resulting pairwise consonant chords.

[^4]
### 1.7 Array representation of tuneable intervals (overtones and undertones)

For the purposes of the following discussion, it is useful to examine an array representation of the tuneable intervals (the element ( $\mathrm{i}, \mathrm{j}$ ) is simply $\mathrm{i} / \mathrm{j}$ in lowest terms):

Table 4

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1/1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 2/1 | 1/1 | 2/3 | 1/2 | 2/5 | 1/3 | 2/7 | 1/4 | 2/9 | 1/5 | 2/11 | 1/6 | 2/13 | 1/7 | 2/15 | 1/8 |  |  |  |  |  |  |  |  |  |  |  |  |
| 3 | 3/1 | 3/2 | 1/1 | 3/4 | 3/5 | 1/2 | 3/7 | 3/8 | 1/3 | 3/10 | 3/11 | 1/4 | 3/13 | 3/14 | 1/5 | 3/16 | 3/17 | 1/6 | 3/19 | 3/20 | 1/7 | 3/22 | 3/23 | 1/8 |  |  |  |  |
| 4 | 4/1 | 2/1 | 4/3 | 1/1 | 4/5 | 2/3 | 4/7 | 1/2 | 4/9 | 2/5 | 4/11 | 1/3 | 4/13 | 2/7 | 4/15 | 1/4 | 4/17 | 2/9 | 4/19 | 1/5 | 4/21 | 2/11 | 4/23 | 1/6 | 4/25 | 2/13 | 4/27 | 1/7 |
| 5 | 5/1 | 5/2 | 5/3 | 5/4 | 1/1 | 5/6 | 5/7 | 5/8 | 5/9 | 1/2 | 5/11 | 5/12 | 5/13 | 5/14 | 1/3 | 5/16 | 5/17 | 5/18 | 5/19 | 1/4 | 5/21 | 5/22 | 5/23 | 5/24 | 1/5 | 5/26 |  | 5/28 |
| 6 | 6/1 | 3/1 | 2/1 | 3/2 | 6/5 | 1/1 | 6/7 | 3/4 | 2/3 | 3/5 | 6/11 | 1/2 | 6/13 | 3/7 | 2/5 | 3/8 | 6/17 | 1/3 | 6/19 | 3/10 | 2/7 | 3/11 | 6/23 | 1/4 | 6/25 | 3/13 | 2/9 | 3/14 |
| 7 | 7/1 | 7/2 | 7/3 | 7/4 | 7/5 | 7/6 | 1/1 | 7/8 | 7/9 | 7/10 | 7/11 | 7/12 | 7/13 | 1/2 |  |  | 7/17 | 7/18 |  | 7/20 | 1/3 | 7/22 |  | 7/24 |  |  | 7/27 | 1/4 |
| 8 | 8/1 | 4/1 | 8/3 | 2/1 | 8/5 | 4/3 | 8/7 | 1/1 |  | 4/5 | 8/11 | 2/3 | 8/13 | 4/7 | 8/15 | 1/2 |  | 4/9 | 8/19 | 2/5 |  | 4/11 | 8/23 | 1/3 | 8/25 | 4/13 | 8/27 | 2/7 |
| 9 |  | 9/2 | 3/1 | 9/4 | 9/5 | 3/2 | 9/7 |  | 1/1 |  | 9/11 | 3/4 | 9/13 | 9/14 | 3/5 |  |  | 1/2 |  |  | 3/7 |  |  | 3/8 |  |  | 1/3 | 9/28 |
| 10 |  | 5/1 | 10/3 | 5/2 | 2/1 | 5/3 | 10/7 | 5/4 |  | 1/1 |  | 5/6 | 10/13 | 5/7 | 2/3 | 5/8 |  | 5/9 |  | 1/2 |  | 5/11 |  | 5/12 | 2/5 | 5/13 |  | 5/14 |
| 11 |  | 11/2 | 11/3 | 11/4 | 11/5 | 11/6 | 11/7 | 11/8 | 11/9 |  | 1/1 |  |  |  |  | 11/16 |  |  |  |  |  | 1/2 |  |  |  |  |  |  |
| 12 |  | 6/1 | 4/1 | 3/1 | 12/5 | 2/1 | 12/7 | 3/2 | 4/3 | 6/5 |  | 1/1 |  | 6/7 | 4/5 | 3/4 |  | 2/3 |  | 3/5 | 4/7 | 6/11 | 12/23 | 1/2 |  | 6/13 | 4/9 | 3/7 |
| 13 |  | 13/2 | 13/3 | 13/4 | 13/5 | 13/6 | 13/7 | 13/8 | 13/9 | 13/10 |  |  | 1/1 |  |  |  |  |  |  |  |  |  |  |  |  | 1/2 |  |  |
| 14 |  | 7/1 | 14/3 | 7/2 | 14/5 | 7/3 | 2/1 | 7/4 | 14/9 | 7/5 |  | 7/6 |  | 1/1 |  | 7/8 |  | 7/9 |  | 7/10 | 2/3 | 7/11 |  | 7/12 |  | 7/13 |  | 1/2 |
| 15 |  | 15/2 | 5/1 | 15/4 | 3/1 | 5/2 |  | 15/8 | 5/3 | 3/2 |  | 5/4 |  |  | 1/1 |  |  | 5/6 |  | 3/4 | 5/7 |  |  | 5/8 | 3/5 |  | 5/9 |  |
| 16 |  | 8/1 | 16/3 | 4/1 | 16/5 | 8/3 |  | 2/1 |  | 8/5 | 16/11 | 4/3 |  | 8/7 |  | 1/1 |  |  |  | 4/5 |  | 8/11 |  | 2/3 |  | 8/13 |  | 4/7 |
| 17 |  |  | 17/3 | 17/4 | 17/5 | 17/6 | 17/7 |  |  |  |  |  |  |  |  |  | 1/1 |  |  |  |  |  |  |  |  |  |  |  |
| 18 |  |  | 6/1 | 9/2 | 18/5 | 3/1 | 18/7 | 9/4 | 2/1 | 9/5 |  | 3/2 |  | 9/7 | 6/5 |  |  | 1/1 |  |  | 6/7 | 9/11 |  | 3/4 |  | 9/13 | 2/3 | 9/14 |
| 19 |  |  | 19/3 | 19/4 | 19/5 | 19/6 |  | 19/8 |  |  |  |  |  |  |  |  |  |  | 1/1 |  |  |  |  |  |  |  |  |  |
| 20 |  |  | 20/3 | 5/1 | 4/1 | 10/3 | 20/7 | 5/2 |  | 2/1 |  | 5/3 |  | 10/7 | 4/3 | 5/4 |  |  |  | 1/1 |  |  |  | 5/6 | 4/5 | 10/13 |  | 5/7 |
| 21 |  |  | 7/1 | 21/4 | 21/5 | 7/2 | 3/1 |  | 7/3 |  |  | 7/4 |  | 3/2 | 7/5 |  |  | 7/6 |  |  | 1/1 |  |  | 7/8 |  |  | 7/9 | 3/4 |
| 22 |  |  | 22/3 | 11/2 | 22/5 | 11/3 | 22/7 | 11/4 |  | 11/5 | 2/1 | 11/6 |  | 11/7 |  | 11/8 |  | 11/9 |  |  |  | 1/1 |  |  |  |  |  |  |
| 23 |  |  | 23/3 | 23/4 | 23/5 | 23/6 |  | 23/8 |  |  |  | 23/12 |  |  |  |  |  |  |  |  |  |  | 1/1 |  |  |  |  |  |
| 24 |  |  | 8/1 | 6/1 | 24/5 | 4/1 | 24/7 | 3/1 | 8/3 | 12/5 |  | 2/1 |  | 12/7 | 8/5 | 3/2 |  | 4/3 |  | 6/5 | 8/7 |  |  | 1/1 |  |  |  | 6/7 |
| 25 |  |  |  | 25/4 | 5/1 | 25/6 |  | 25/8 |  | 5/2 |  |  |  |  | 5/3 |  |  |  |  | 5/4 |  |  |  |  | 1/1 |  |  |  |
| 26 |  |  |  | 13/2 | 26/5 | 13/3 |  | 13/4 |  | 13/5 |  | 13/6 | 2/1 | 13/7 |  | 13/8 |  | 13/9 |  | 13/10 |  |  |  |  |  | 1/1 |  |  |
| 27 |  |  |  | 27/4 |  | 9/2 | 27/7 | 27/8 | 3/1 |  |  | 9/4 |  |  | 9/5 |  |  | 3/2 |  |  | 9/7 |  |  |  |  |  | 1/1 |  |
| 28 |  |  |  | 7/1 | 28/5 | 14/3 | 4/1 | 7/2 | 28/9 | 14/5 |  | 7/3 |  | 2/1 |  | 7/4 |  | 14/9 |  | 7/5 | 4/3 |  |  | 7/6 |  |  |  | 1/1 |

The ratios in this representation may be regarded as a set of pitches in relation to $1 / 1$. Note that each column is a set of overtones (a harmonic series) and each row is a set of undertones (a subharmonic series). The interlocking structure of overtones and undertones is fundamental to understanding how acoustic instruments generate subsets of the total harmonic space and how these subsets may be traversed by perceptible harmonic intervals.

String instruments, for example, produce overtones (harmonics) on each string at nodal points dividing the string length into integer parts. A string vibrating at frequency F has partials with frequencies $\{F, 2 F, 3 F, \ldots\}$. The $\mathrm{n}^{\text {th }}$ partial (with frequency nF ) divides a string of unit length into $n$ parts, generating ( $n-1$ ) possible ${ }^{10}$ nodal points with associated string lengths $\{1 / n, 2 / n, \ldots,(n-1) / n\}$. Because string length is inversely proportional to frequency ${ }^{11}$, the possible nodal points for the $\mathrm{n}^{\text {th }}$ partial have frequencies $\{\mathrm{nF},(\mathrm{n} / 2) \mathrm{F}, \ldots,(\mathrm{n} /(\mathrm{n}-1)) \mathrm{F}\}$. This is a

[^5]set of undertones (subharmonics) of the $\mathrm{n}^{\text {th }}$ partial. Thus, the array of pitches consisting of possible nodal points for each partial is a set of interlocking overtones and undertones generated by the string fundamental frequency $F$. (Consider the ratios below the diagonal in the array above.)

## 2. MICROTONALITY ON VALVED BRASS INSTRUMENTS

### 2.1 The combined range of the horn, trombone and tuba

The combined range of the horn, trombone and tuba is 5 octaves, descending from F5 to F0 ( 704 Hz and 22 Hz respectively assuming A4 $=440 \mathrm{~Hz}$ ). As the very lowest pitches of the tuba lie at the border between pitch and noise, and since the timbre of the very highest horn and trombone notes becomes increasingly thin, the practical range in terms of tuneability extends over four-and-a-half octaves from D5 down to A0.

### 2.2 The purpose of the valve

Before the introduction of the valve in the early 19th century, all of the pitches available on natural horns were derived from overtones of a single length of tubing. Various lengths of tubing known as crooks were used to transpose the instrument into different keys, and each length also gave a distinctive colour to the sound. The natural pitches of the overtone series could be altered by hand-stopping, a technique which inevitably had a strong influence on the timbre of each note. Valves were invented in order to make natural horns fully chromatic and enable a homogenous sound throughout all pitches.

A valved horn may also be regarded as a collection of natural horns, each with its own overtone row. By tuning the valve-slides in whole number proportions to one another, these overtone rows become harmonically related and a network of natural intervals emerges. Pitches differing by very small melodic steps in this microtonal gamut may be played on widely varying lengths of tubing, creating alternations of tone-colour reminiscent of natural horn technique.

### 2.3 Utonal valve tuning

If the second valve is tuned to add $1 / 15$ of the open horn length, the resulting interval when the valve is depressed is the semitone 15:16. Tuning the first valve to $2 / 15$ of the open horn length results in the interval 15:17. Combining these valves creates the minor third 15:18. If the third valve is tuned to add $4 / 15$ of the open horn length the resulting interval is 15:19. Combining the second and third valves produces 15:20; combining the first and third valves produces 15:21, and a combination of all three valves $(1+2+3)$ results in 15:22.

| Valves | Proportion added | Ratio to open horn |
| :---: | :---: | :---: |
| 2 | 1/15 | 15:16 |
| 1 | 2/15 | 15:17 |
| 1+2 | 3/15 | $15: 18=5: 6$ |
| 3 | 4/15 | 15:19 |
| 2+3 | $5 / 15=1 / 3$ | $15: 20=3: 4$ |
| 1+3 | $6 / 15=2 / 5$ | $15: 21=5: 7$ |
| 1+2+3 | 7/15 | 15:22 |

The valve combinations thus form an undertone row descending from 15 through 22; in the just intonation terminology proposed by Harry Partch it is an utonal tuning of the valves.

In the 1970s the German music theorist Martin Vogel proposed such a tuning method for valved brass instruments ${ }^{12}$. In his system the second valve may be tuned to $1 / 14,1 / 15,1 / 16$, or $1 / 18$ of the overall tube length. The instruments are equipped with a trigger system enabling the valve-slides to be simultaneously adjusted during performance whilst maintaining their proportion to one another $-(1 / n: 2 / n: 4 / n)$ where $n=14,15,16$ or 18 . Note that the set of all valve-combinations tuned in this 'binary' fashion is $\{1 / \mathrm{n}, 2 / \mathrm{n}, \ldots 7 / \mathrm{n}\}$. Adjusting between the four different tuning systems makes extremely virtuosic demands on the player. The slide action is mechanically intricate, and there is no possibility of easily changing the instrument back to a conventional valve tuning.

### 2.4 Conventional tuning of the valves

The first three valves of a brass instrument lower the pitch by a tone, semitone and a minor third. Combinations of the three valves create a descending chromatic scale over a diminished 5th: for example, $\mathrm{F}-\mathrm{E}-\mathrm{Eb}-\mathrm{D}-\mathrm{Db}-\mathrm{C}-\mathrm{B}$.

Let $H$ be the length of the open horn, and $V_{1}$ and $V_{2}$ be the lengths of tubing added by two of the valves. Then, since $H<\left(H+V_{2}\right)$,

```
\(1 / \mathrm{H}>1 /\left(\mathrm{H}+\mathrm{V}_{2}\right)\)
\(\mathrm{V}_{1} / \mathrm{H}>\mathrm{V}_{1} /\left(\mathrm{H}+\mathrm{V}_{2}\right)\)
\(1+\left(\mathrm{V}_{1} / \mathrm{H}\right)>1+\left(\mathrm{V}_{1} /\left(\mathrm{H}+\mathrm{V}_{2}\right)\right)\)
\((\mathrm{H} / \mathrm{H})+\left(\mathrm{V}_{1} / \mathrm{H}\right)>\left(\mathrm{H}+\mathrm{V}_{2}\right) /\left(\mathrm{H}+\mathrm{V}_{2}\right)+\left(\mathrm{V}_{1} /\left(\mathrm{H}+\mathrm{V}_{2}\right)\right)\)
and thus \(\left(\mathrm{H}+\mathrm{V}_{1}\right) / \mathrm{H}>\left(\mathrm{H}+\mathrm{V}_{2}+\mathrm{V}_{1}\right) /\left(\mathrm{H}+\mathrm{V}_{2}\right)\).
```

Therefore, each valve lowers the pitch by a larger interval when used on its own than when used in combination with another valve. In the descending scale above, the semitone between F and E is larger than that from C to B , though both steps are brought about by depressing the second valve.

Each of the three valves may be tuned individually so as to lower the pitch by a tempered tone, semitone, and minor third, but their combinations will not coincide with tempered tuning. Also, since brass instruments are played by sounding the overtone series, many of the available pitches will deviate from Equal Temperament even when the fundamental is tuned to a tempered pitch.

### 2.5 Rational approximation of the tempered tuning

The closest approximation to Equal Temperament on the horn is attained by the following valve tuning. Note that the four traditionally employed combinations ( $1,2,1+2,2+3$ ) produce tempered fundamentals within a tolerance of 5 cents.

| Valves | Proportion added | Ratio to open horn | Lowers by (cents) |
| :---: | :---: | :---: | :---: |
| 1 | $1 / 8=10 / 80$ | 8:9 | 203.9ф |
| 2 | $1 / 16=5 / 80$ | 16:17 | 105¢ |
| 1+2 | $3 / 16=15 / 80$ | 16:19 | 297.5¢ |
| 3 | $1 / 5=16 / 80$ | 5:6 | 315.6 ¢ |
| 2+3 | 21/80 | 101:80 | 403.5¢ |
| 1+3 | $13 / 40=26 / 80$ | 53:40 | 487.2¢ |
| 1+2+3 | 31/80 | 111:80 | 567.0¢ |

[^6]The tuba may be tuned similarly. Fundamentals near tempered pitches are indicated in boldface.

## Table 5

| Valves | Proportion added | Ratio to open horn | Lowers by (cents) |
| :---: | :---: | :---: | :---: |
| 2 | 1/16 = 15/240 | 17/16 | 105 |
| 6 | $1 / 12=20 / 240$ | 13/12 | 138.6 |
| 1 | $1 / 8=30 / 240$ | 9/8 | 203.9 |
| 2+6 | $7 / 48=35 / 240$ | 55/48 | 235.7 |
| 5 | $1 / 6=40 / 240$ | 7/6 | 266.9 |
| 1+2 | 3/16 = 45/240 | 19/16 | 297.5 |
| 3 | $1 / 5=48 / 240$ | 6/5 | 315.6 |
| 1+6 | $5 / 24=50 / 240$ | 29/24 | 327.6 |
| 2+5 | $11 / 48=55 / 240$ | 59/48 | 357.2 |
| 5+6 | $1 / 4=60 / 240$ | 5/4 | 386.3 |
| 2+3 | 21/80 $=63 / 240$ | 101/80 | 403.5 |
| 1+2+6 | $13 / 48=65 / 240$ | 61/48 | 414.9 |
| 3+6 | $17 / 60=68 / 240$ | 77/60 | 431.9 |
| 1+5 | $7 / 24=70 / 240$ | 31/24 | 443.1 |
| $2+5+6$ | $5 / 16=75 / 240$ | 21/16 | 470.8 |
| 1+3 | $13 / 40=78 / 240$ | 53/40 | 487.2 |
| 4 | 1/3 = 80/240 | 4/3 | 498 |
| 2+3+6 | $83 / 240=83 / 240$ | 323/240 | 514.2 |
| $1+2+5$ | $17 / 48=85 / 240$ | 65/48 | 524.9 |
| $3+5$ | $11 / 30=88 / 240$ | 41/30 | 540.8 |
| 1+5+6 | $3 / 8=90 / 240$ | 11/8 | 551.3 |
| $1+2+3$ | $31 / 80=93 / 240$ | 111/80 | 567 |
| 2+4 | 19/48 = 95/240 | 67/48 | 577.4 |
| 1+3+6 | $49 / 120=98 / 240$ | 169/120 | 592.8 |
| 4+6 | 5/12 = 100/240 | 17/12 | 603 |
| $2+3+5$ | 103/240 $=103 / 240$ | 343/240 | 618.2 |
| $1+2+5+6$ | $7 / 16=105 / 240$ | 23/16 | 628.3 |
| 3+5+6 | 9/20 $=108 / 240$ | 29/20 | 643.3 |
| 1+4 | $11 / 24=110 / 240$ | 35/24 | 653.2 |
| $1+2+3+6$ | 113/240 $=113 / 240$ | 353/240 | 668 |
| 2+4+6 | 23/48 $=115 / 240$ | 71/48 | 677.7 |
| $1+3+5$ | 59/120 = 118/240 | 179/120 | 692.3 |
| 4+5 | $\mathbf{1 / 2} \mathbf{= 1 2 0 / 2 4 0}$ | 3/2 | 702 |
| $2+3+5+6$ | $41 / 80=123 / 240$ | 121/80 | 716.3 |
| 1+2+4 | 25/48 $=125 / 240$ | 73/48 | 725.8 |
| $3+4$ | 8/15 = 128/240 | 23/15 | 740 |
| 1+4+6 | $13 / 24=130 / 240$ | 37/24 | 749.4 |
| $1+2+3+5$ | $133 / 240=133 / 240$ | 373/240 | 763.4 |
| $2+4+5$ | 9/16 = 135/240 | 25/16 | 772.6 |
| $1+3+5+6$ | $23 / 40=138 / 240$ | 63/40 | 786.4 |
| 4+5+6 | 7/12 = 140/240 | 19/12 | 795.6 |
| 2+3+4 | $143 / 240=143 / 240$ | 383/240 | 809.2 |
| $1+2+4+6$ | 29/48 $=145 / 240$ | 77/48 | 818.2 |
| $3+4+6$ | $37 / 60=148 / 240$ | 97/60 | 831.6 |
| $1+4+5$ | $5 / 8=150 / 240$ | 13/8 | 840.5 |


| $1+2+3+5+6$ | $51 / 80=153 / 240$ | $131 / 80$ | 853.8 |
| :--- | :--- | :--- | :--- |
| $2+4+5+6$ | $31 / 48=155 / 240$ | $79 / 48$ | 862.6 |
| $1+3+4$ | $79 / 120=158 / 240$ | $199 / 120$ | 875.7 |
| $\mathbf{2 + 3 + 4 + 6}$ | $\mathbf{1 6 3 / 2 4 0}=\mathbf{1 6 3 / 2 4 0}$ | $\mathbf{4 0 3 / 2 4 0}$ | $\mathbf{8 9 7 . 3}$ |
| $1+2+4+5$ | $11 / 16=165 / 240$ | $27 / 16$ | 905.9 |
| $3+4+5$ | $7 / 10=168 / 240$ | $17 / 10$ | 918.6 |
| $1+4+5+6$ | $17 / 24=170 / 240$ | $41 / 24$ | 927.1 |
| $1+2+3+4$ | $173 / 240=173 / 240$ | $413 / 240$ | 939.7 |
| $1+3+4+6$ | $89 / 120=178 / 240$ | $209 / 120$ | 960.6 |
| $2+3+4+5$ | $61 / 80=183 / 240$ | $141 / 80$ | 981.1 |
| $1+2+4+5+6$ | $37 / 48=185 / 240$ | $85 / 48$ | 989.3 |
| $\mathbf{3 + 4 + 5 + 6}$ | $\mathbf{4 7 / 6 0}=\mathbf{1 8 8 / 2 4 0}$ | $\mathbf{1 0 7 / 6 0}$ | $\mathbf{1 0 0 1 . 5}$ |
| $1+2+3+4+6$ | $193 / 240=193 / 240$ | $433 / 240$ | 1021.6 |
| $1+3+4+5$ | $33 / 40=198 / 240$ | $73 / 40$ | 1041.5 |
| $2+3+4+5+6$ | $203 / 240=203 / 240$ | $443 / 240$ | 1061.1 |
| $\mathbf{1 + 2 + 3 + 4 + 5}$ | $\mathbf{7 1 / 8 0}=\mathbf{2 1 3 / 2 4 0}$ | $\mathbf{1 5 1 / 8 0}$ | $\mathbf{1 0 9 9 . 8}$ |
| $1+3+4+5+6$ | $109 / 120=218 / 240$ | $229 / 120$ | 1118.8 |
| $1+2+3+4+5+6$ | $233 / 240=233 / 240$ | $473 / 240$ | 1174.6 |

The complete microtonal pitch set available in this tuning consists of the harmonic series over these fundamentals. In the following example, commonly played tuba fingerings for the chromatic scale are compared with their theoretical intonation (see note below about the notation of fingerings). Alternate fingerings for the four notes which deviate from tempered tuning are marked below the standard fingerings. Since the common denominator used is 240 , this tuning is referred to as the 240-utonal tuba tuning.

## Table 6

## 240-Utonal Tuba: Chromatic Scale With Conventional Fingerings

Tempered F


The notation of fingerings is adapted from woodwind notation. For the tuba: the column of four circles indicates valves $1-4$, operated by the right hand, and the column of two circles valves 5 and 6 , operated by the left hand. For the horn: the column of three circles indicates valves $1-3$, and the single circle the thumb valve which switches between the Bb and F horn. All horn valves are operated by the right hand. Note that these fingerings are written for tubas on which the fifth and sixth valves are a large tone and semitone respectively and for horns on which Bb-side is played with the thumb valve depressed. For instruments with these valves reversed, fingerings must be adapted accordingly.

### 2.6 Utonal horn tunings

Tuning the valve slides according to an undertone row incorporates the valve combinations and overtone rows as integral parts of an expanded microtonal tuning system. Of the four semitone tunings mentioned earlier ( $14: 15,15: 16,16: 17$, and $18: 19$ ), only $15: 16$ and 16:17 may be arrived at by adjusting the valve-slides of a standard horn, without making physical alterations to the instrument itself.

The two resulting sets of undertones are notated below with valve fingerings using the Extended Helmholtz-Ellis JI Pitch Notation.

Table 7

## Double Horn 15 and 16: Fundamental Pitches



In our work to date as composers, we have prioritized the $1 / 15$ over the $1 / 16$ tuning because it offers more musically useful pitches when combining brass instruments with strings (note that the Bb and F raised by a comma have D and A as fifth partials, matching open strings of a violin, viola, cello or bass). Rather than using a trigger system for retuning the slides while playing, we apply this single fixed utonal tuning to a modern double horn in Bb and F . However, we do incorporate Vogel's proposal to make custom longer slides for the third valve ( $4 / 15$ instead of $3 / 15$ of the open horn lengths), eliminating the synonym $1+2=3$.

The $1 / 15$ tuning is called the 45-utonal horn tuning, because all resulting undertones on both sides of the instrument can be represented as part of a single series:

Bb-horn
45:48:51:54:57:60:63:66
F-horn
60:64:68:72:76:80:84:88
The following table shows the available pitches on a double horn tuned in this manner, organized as sixteen harmonic series over the sixteen undertone fundamentals produced by the set of all valve combinations.

Table 8 (overleaf)

Double Horn in Bb and F / Bb-Horn : Tuning in Just Intonation
valves 1, 2, $3^{*}$ (custom-built) are tuned to rational proportions (2/15, 1/15, 4/15) of the respective open horn's length producing, in various combinations, two Utonal Series of fundamental pitches with wavelengths in the proportions $15: 16: 17: 18: 19: 20: 21: 22$ with the two horns tuned a just perfect fourth (3:4) apart
notated AT SOUNDING PITCH


Double Horn in Bb and F / F-Horn : Tuning in Just Intonation
valves 1, 2, $3^{*}$ (custom-built) are tuned to rational proportions (2/15, $1 / 15,4 / 15$ ) of the respective open horn's length producing, in various combinations, two Utonal Series of fundamental pitches with wavelengths in the proportions $15: 16: 17: 18: 19: 20: 21: 22$ with the two horns tuned a just perfect fourth (3:4) apart * Alternately, the normal 3 rd valve may be used, tuned to the proportion $3 / 15$, so that the valve combination $1+2$ and 3 are synonymous.
In this case, the undertone tuning " 22 " will not be available, and the fingerings indicated for undertones $19-22$ apply instead to $18-21$ In this case, the undertone tuning "22" will not be available, and the fingerings indicated for undertones 19-22 apply instead to 18-21 accidentals designed by Marc Sabat and Wolfgang von Schweinitz, 2004


The preceding table indicates pitches based on a theoretical model. To evaluate the accuracy of this model, sensors were constructed for each valve and the horn and tuba were connected to a computer. A MAX/MSP program generated exact overtone rows over each theoretical fundamental using sinewaves. The instrumental sounds were ring-modulated with these sinewaves to test the accuracy of the valve combinations. It was possible to determine valve-slide positions which allowed the desired pitches to be produced within a very acceptable tolerance. As expected, naturally occurring inharmonicities demand that the players sometimes make fine corrections with their lips. However, once measured, the valveslide positions can be set before playing and the results remain consistent once the instruments have been warmed up.

### 2.7 Possible extensions of the horn tuning

In order to further extend the pitch-gamut of the horn, players may employ half-valve and hand-stopping techniques. To gain even more useful pitches it is necessary to make alterations to the physical structure of the horn.

Many horns come equipped with a stopping valve, designed to lower the pitch by a semitone. By altering this to produce the quartertone ratio $30: 31$, the set of available fundamentals becomes 30:31:32:33:34:35:36:37:38:39:40:41:42:43:44:45. Notably, this includes the 13limit undertone 39, and would link the horn to more of the higher-prime tuba fundamentals discussed below.

### 2.8 Utonal tuba tunings

In contrast to the horn, tuba designs are anything but standardized. Bass tubas may be pitched in F or Eb , contrabass tubas in C or Bb . Instruments may have between 3 and 6 valves, and a variety of compensating systems are available, designed to adjust the valve combinations closer to tempered tuning. For the purposes of this study the 6 -valved F tuba has been chosen, as it is the most closely related to the horn and offers more microtonal possibilities than tubas with less valves.

The length of the $F$ tuba is identical to that of the $F$ horn, and the first three valves produce the same intervals. The fourth valve lowers the fundamental by a perfect fourth to C . The fifth and sixth valves are intended as tone and semitone valves in proportion to the length of the tuba with the fourth valve depressed.

This 6 -valve system may also be tuned utonally. Following the example of the horn, the first three valves may be tuned to produce 15:16, 15:17, and 15:18, bringing about an undertone series from 15 to 21 . The fourth valve may be regarded as transposing the $F$ tuba into a $C$ tuba, in effect creating a double tuba in F and C . The fifth and sixth valves act as the first and second valves of this C tuba, and may be tuned to produce the intervals 15:16 and 15:17.

| Valves | Proportion added | $\underline{\text { Ratio to open horn }}$ |
| :---: | :---: | :---: |
| $0=F$ tuba | none | 15:15 |
| 1 | $2 / 15=6 / 45$ | 15:17 |
| 2 | $1 / 15=3 / 45$ | 15:16 |
| 1+2 [= 3] | $3 / 15=9 / 45$ | 15:18 |
| $4=C$ tuba | $1 / 3=15 / 45$ | $(3: 4)(15: 15)$ |
| 5 | $(2 / 15)(4 / 3)=8 / 45$ | (3:4)(15:17) |
| 6 | $(1 / 15)(4 / 3)=4 / 45$ | (3:4)(15:16) |
| $5+6[=2+3]$ | $(3 / 15)(4 / 3)=12 / 45$ | $(3: 4)(15: 18)$ |

Since $2+3$ is synonymous with $5+6$, this combination acts a "third valve" for the $C$ tuba. Combinations of $2+3$ with the fifth and sixth valves thus extend its undertone row down to 21 .

Table 9

## Double Tuba in F and C : Fundamental Pitches



Calculating all 64 possible valve combinations results in the following table of pitches, with 40 distinct fundamentals descending one octave from F1 to F0, tuned as undertones between 45 and 90 , called the 45-utonal tuba tuning.

Table 10 (overleaf)

6-Valve F Tuba Tuning in Just Intonation
valves 1, 2, 3, 4, 5, 6 tuned to rational proportions ( $6 / 45,3 / 45,9 / 45,15 / 45,8 / 45,4 / 45$ ) of the open tuba's length producing, in various combinations, a Utonal Series of fundamental pitches with wavelengths in the proportions $45: 48: 49: 51: 52: 53: 54: 55: 56: 57: 58: 59: 60$ $61: 62: 63: 64: 65: 66: 67: 68: 69: 70: 71: 72: 73: 74: 75: 76: 77: 78: 79: 80: 81: 82: 83: 84: 86: 87: 90$





This table effectively multiplies the 15 -utonal tuning by 3 , dividing each semitone step into three unequal parts. An analogous procedure might be considered with higher multiples of $15: 16$. If the second valve is considered to be producing the undertone relationship 60:64, the sixth valve may be tuned to $60: 65$ and the fifth valve $60: 70$. The resulting undertone row would now descend from 60 to 119, dividing the semitones into four unequal parts (60-utonal tuba tuning). For example, 16:17 may be divided into three parts 48:49:50:51 and into four parts 64:65:66:67:68. Because no valve adds a unit proportion (1/45 or 1/60 of the length), both tunings have gaps in their undertone rows. In the 45-utonal table the missing undertones are $46,47,50,85,88,89$. The 60 -utonal table is missing $61,62,63,66,67,71$, 108, 112, 113, 116, 117, 118.

Complete undertone rows may be generated by valves in the proportions 1:2:4:8:16:32. This 'binary' tuning could be realised by shortening the sixth and fifth valves to $1 / 60$ and $2 / 60$ of the open tube, leaving the second and first valves tuned to $4 / 60$ and $8 / 60$, and extending the third and fourth valves to $16 / 60$ and 32/60:

| Valve |  | Proportion added |  |
| :--- | :--- | :--- | :--- |
|  |  |  | Ratio to open horn |
|  |  |  | $15=8 / 60$ |
| 3 | $1 / 15=4 / 60$ |  | $60: 68$ |
| 3 |  | $4 / 15=16 / 60$ |  |
| 4 | $8 / 15=32 / 60$ |  | $60: 76$ |
| 5 | $1 / 30=2 / 60$ | $60: 92$ |  |
| 6 | $1 / 60$ | $60: 62$ |  |
|  |  |  | $60: 61$ |

This would complete the 60 -utonal tuning, giving 64 unique undertone fundamentals descending from 60 through 123. The addition of a trigger attached to the main slide could further extend the binary principle to include $1 / 120$ and $1 / 240$ of the open tube. This utopian microtonal tuba would include as subsets all tunings discussed above (240, 45, 60).

### 2.9 Comparison of the horn, tuba and trombone

In the case of the double horn, two independent instruments are combined in one: a Bb-horn with three valves and an F-horn with three valves. A thumb trigger valve redirects the airflow from one side of the instrument to the other. There are therefore six separate valve slides, operated by three valves. In this way, any valve combination on the F-side of the horn is always tuned a perfect fourth below the same valve combination on the Bb -side.

The tenorbass trombone consists of two instruments as well, the tenor trombone in Bb and the bass trombone in $F$ (tuned to the same fundamental pitches as the two sides of the open horn). A slide may be extended continuously from 0 cm to around 60 cm , adding between 0 and 120 cm to the open tube. This allows the Bb -side to descend in pitch by approximately $600 \phi$. Switching between the two sides is accomplished similarly to the horn, with a thumb trigger valve. In this case, however, the valve adds a fixed length of tubing, rather than redirecting the airflow to an independent system. Thus, the interval between the two sides decreases as the slide position increases.

In the case of the tuba, a single length of tubing tuned to $F$ (the same fundamental as the horn and trombone F-sides) is extended with combinations of six valves. The first three are identical to the horn, and the fourth valve is tuned to lower the pitch by a perfect fourth ( $1 / 3$ of the open tube length, producing a $3: 4$ ratio). The fourth valve on the tuba, when used to alternate between otherwise identical valve combinations, functions identically to the trombone trigger valve, lowering by decreasing amounts as the length of tubing increases.

### 2.10 Slide division of the tenorbass trombone

Traditionally, trombone players learn seven positions on the Bb -side of their instrument, corresponding to the tempered semitones descending from Bb1 down to E1. On the F-side, there are six positions, ranging from a somewhat low F1 descending down to C1. The chromatic fundamental BO is not present. The trigger valve-slide is tuned to an interval between a fourth and tritone, allowing the slide ${ }^{13}$ to reach down to C 1 at full extension. The trigger valve-slide has a great degree of flexibility: in the first position it may be tuned as narrow as $1 / 3$ of the Bb side, producing a 3:4 relationship between the two sides (498 $\phi$ ). It may be pulled out to $3 / 8$ of the Bb side, producing an $8: 11$ relationship (551.3申), and in some cases to as much as $2 / 5$ of the Bb side, producing a $5: 7$ relationship ( $582.5 \phi$ ).

Each of these rational possibilities suggests an undertone division of the slide into equidistant steps. Note that a slide extension of $n$ centimeters extends the overall tube length by 2 n . Thus, an extension of 48 cm lowers the Bb comma up on the Bb side by a perfect fourth to F comma up (from 288 cm to 384 cm ). If the distance between Bb and F is divided into $n$ integer parts, a descending utonal series $\{3 n, 3 n+1, . ., 4 n\}$ is obtained. In the case of a trigger valve-slide tuned to $1 / 3$ of the Bb side, the corresponding points on the F -side will then be tuned to $\{4 n, 4 n+1, \ldots, 5 n\}$ in the same utonal series. Depending on the choice of $n$, the series on both sides may be extended as far as the slide allows. Since its maximum extension is 60 cm , and the step size is ( $48 / \mathrm{n}$ ), the maximum number of steps is the greatest integer less than or equal to ( $5 \mathrm{n} / 4$ ).

To maximize congruence with the available fundamentals of the horn and tuba, a division of the 48 cm length into 45 parts (steps of $1,1 \mathrm{~cm}$ ) is used. This produces steps ranging from 135:136 (12.8 $)$ at the top of the Bb-side to $235: 236(7.4 \phi)$ at the bottom of the F-side. Roughly described, it divides each conventional position into nine to ten parts. The following horn and tuba fundamentals may be played by the trombone at unison or one octave higher:

Bb-Horn:
(unison) 45, 48, 51, 54, 57, 60, 63, 66
F-Horn:
(unison) 60, 64, 68, 72, 76
( 80,84 and 88 are below the range of the trombone's F -side)
Tuba:
(unison) 45, 48, 49, 51, 52, 53, 54, 55, 56, 57, 58, 59
( 60 through 67 are below the range of the trombone's F-side)
(octave higher) $68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,86,87,90$

Of course, it is clear that trombonists can technically reach any fundamental within the available range ( C through Bb ). Nevertheless, equal division of the slide gives a conceptual framework for combining the trombone with the utonal tunings of the horn and tuba. The following table indicates the first twelve steps of the trombone 45-division. Pitches with boxed text above indicate corresponding horn and tuba undertones. Bracketed pitches are enharmonic simplifications (within a 5-cent tolerance range) of frequency ratios including primes above 13.

## Table 11 (overleaf)

[^7]


### 2.11 Two musical examples

Wonderful Scatter (2005) is a composition by Marc Sabat for tuba in the 45-utonal tuning, computer and loudspeaker (see score example 1 below). Sensors are installed under each of the six valves, continually measuring their positions. Movements produced by the tubist's fingers (and thus information about the valve-slide lengths being used at any given time) are transmitted in real-time across an ethernet cable to the computer. The sensor mechanism was developed in Berlin by Sukandar Kartadinata.

The tubist plays all available pitches in the first four octaves of the instrument's range, organized in an ascending microtonal gamut. The lengths of notes are inversely proportional to the respective tube lengths (a tube length of 45 is played for one pulse at 45 bpm and a tube length of 90 is played for one pulse at 90 bpm ). Notes are separated in time by durations proportional to the melodic distance between them (measured in cents) so that each semitone contains one minute of non-playing pause. After the second, third and fourth octaves, the player repeats the lowest octave (pedal tones).

A MAX/MSP patch analyzes the valve combinations. Corresponding to the 40 possible tubelengths there are 40 loops, and each loop is allocated a duration proportional to its respective fundamental wavelength. Using information from the sensors, each new note is routed to its corresponding loop. Over the course of the performance, the computer acts as a memorysieve, sorting the pitches played into 40 different overtone-series loops, which are activated one-by-one as the tubist changes fingerings. Depending on the lengths of the loops and of the notes played, the resulting textures range from a spectral chord to a chance-determined overtone melody.

A subwoofer and corresponding loudspeaker amplify the output of the computer program, which eliminates higher frequencies (above 352 Hz ), producing a soft, round sound resembling sinewaves. As the music progresses, each new note gradually merges with harmonically related tones and the movement from tone to tone is transformed from a relationship based on melodic distance to various possible relationships of harmonic distance. The title is a literal English translation taken from the Latin Requiem Mass text Tuba mirum, spargens sonum - War-trumpet wonderful, scatter sound.

## Score Example 1

The first 2 pages are to be repeated after each new octave (indicated by signs in the score). Time between notes is measured in quarter notes at 100 bpm , following the rule 100 cents $=1$ minute Blach noteheads are to be recorded as quarter-notes at the indicated tempo (exact bpm may be somewhat exaggerated to delineate the tempo relationships - sempre poco rubato) White noteheads indicate a connected (quasi legato) sequence of quarter-notes, one for each of the notated synonymous valve-combinations.




\%
$+66.3 \quad \Delta=49 \quad \quad \quad=100$
$\xrightarrow{\text { F\#O }}$

$\therefore$ (on the final repeat, after 112 beats release all fingers to produce the fingering "4) without playing a note; slowly fade out loop)


Score Example 2


9


[^8]

Monophones (2006) is a composition by Robin Hayward for horn in the 45-utonal tuning (with normal third valve-slides tuned to $3 / 15$ of the open horn) and 6 -valve F tuba, treated as a double tuba in F and C (see score example 2 above). Pitches are assigned to the horn and tuba based on tetrachords arising from the perfect fourth relationship between the two instruments.

Durations are assigned to the pitches according to which instrument they are played by, their overtone number, and which chart ( $\mathrm{Bb}, \mathrm{F}$ or C ) they occur in.

Table 12


The intervals between horn and tuba are measured according to Euler's gradus suavitatis function outlined in section 1.2 above. Intervals with a consonance grade value $\leq 10$ are distinguished either by repetition or silence.

For example, the first interval in the score, $3 / 1$, which has a consonance grade of 3 , is played 3 times; the interval at the start of the second system, 16/3, having a consonance grade of 7 , is played 7 times. If the number of repetitions exceeds the duration assigned to the second note of the interval, as in bars 45-46, the duration takes priority and the number of repetitions is curtailed. If the second note is shorter than the first note, the interval is followed by a silence lasting the difference between the two durations (see bars 62-65).

## 3. TUNEABLE INTERVALS BETWEEN THE TUBA AND HORN

### 3.1 Calculation and analysis of the available intervals

All possible intervals between the tuba and horn 45 -utonal tunings (tables 5 and 6), have been tabulated to find correspondences and near-correspondences with the set of tuneable intervals. First, the tuba and horn pitches were calculated according to the following conditions:
1.) It is assumed that a player can reach up to the $19^{\text {th }}$ partial on any given length of tubing. To prioritize the even-numbered partials when sorting the resulting lists, these overtones are ordered $\{1,2,3,4,5,6,8,7,9,10,12,11,13,14,15,16,17,18,19\}$. ( 8 precedes 7 and 12 precedes 11.)
2.) The available undertone fundamentals resulting from all valve combinations are determined as a set of integer proportions between 45 and 90 as discussed above.
3.) The set of overtones of each undertone are written as rational numbers (overtone/undertone).
4.) The pitch of the open tube (with no valves depressed) is determined as a ratio in relation to $1 / 1$ (A4). The horn Bb fundamental is thus $2 / 15$, and the tuba F fundamental is $1 / 10$.
5.) Each available pitch is calculated by multiplying the (overtone/undertone) proportions with the frequency ratio of the open tube fundamentals.
6.) The melodic distance from A4 is calculated for each pitch.
7.) The resulting pitches are sorted in rising order.
8.) Pitches outside the playable range of each instrument are eliminated.

In the case of the horn, most of the pedal tones are omitted (with the exception of the two highest, Bb and A ) as well as all pitches above F 5 plus a comma. In the case of the tuba, pitches above C5 plus a comma are omitted. Although the very high tuba pitches between F4 and C5 may be beyond the range of normal playing, they are included because of the "double tuba" model discussed above, which is a perfect fourth below the horn. For an experienced player the pitches are in fact playable, though the timbre in this extreme register becomes increasingly thin and less typical of the normal tuba sound.

This results in 247 horn pitches ranging from A1 to F5, and 1088 tuba pitches from F0 to C5. Many of these pitches are synonyms, which can be produced in several different overtoneundertone combinations and/or valve combinations. To simplify the process of calculating tuneable intervals, the synonyms are eliminated. This leaves 178 distinct horn pitches and 563 distinct tuba pitches, allowing $(178 \cdot 563)=100214$ possible intervals.

An array $\operatorname{TvH}[i, j]$ is constructed, with rows for each tuba pitch $T_{i}$ and columns for each horn pitch $H_{j}$. This array is searched by a computer program in two different ways. The first method calculates the ratios in lowest terms between tuba and horn, $F\left(H_{j}\right) / F\left(T_{i}\right)$. Each ratio is
compared with the set of 183 tuneable interval ratios \{TI\}, and if there is a match it is indicated in the array.

$$
\begin{equation*}
\operatorname{TvH}[i, j]=F\left(H_{j}\right) / F\left(T_{\mathrm{i}}\right) \text { if } \mathrm{F}\left(\mathrm{H}_{\mathrm{j}}\right) / \mathrm{F}\left(\mathrm{~T}_{\mathrm{i}}\right) \in\{\mathrm{T}\} \text { else } \operatorname{TvH}[\mathrm{i}, \mathrm{j}]=\text { blank } \tag{6}
\end{equation*}
$$

In this case, the array contains 4782 tuneable intervals between tuba and horn.
The number of tuneable interval combinations is counted for each row and column of the array. In the case of the horn, each pitch has at least 3 different corresponding tuba pitches. In musical terms, every horn note can generate tuneable tuba melodies. Conversely, however, 245 of the 563 tuba pitches have no corresponding horn notes, and 9 pitches have only one.

This result suggests the need for a second calculation method, which looks for close enharmonic approximations of tuneable intervals. Each interval between tuba and horn is calculated as a melodic distance and is compared with the list of tuneable intervals expressed as melodic distances. The results are compiled in five tables ranging from $\pm 0.5 \phi$ maximum deviation up to $\pm 4.5 \phi$ maximum deviation, at which tolerance all tuba pitches can also be harmonized with two or more distinct horn pitches.

| $\pm 0.5 \phi$ | 5334 tuneable intervals |
| :--- | :--- |
| $\pm 1.5 \phi$ | 8458 tuneable intervals |
| $\pm 2.5 \phi$ | 11860 tuneable intervals |
| $\pm 3.5 \phi$ | 16255 tuneable intervals |
| $\pm 4.5 \phi$ | 21016 tuneable intervals |

These results are filtered into a single table using the following algorithm.
1.) Include all exact ratios.
2.) In a given row, if the number of horn pitches forming a tuneable interval is 0 or 1 (i.e., if melodic movement is not possible in relation to a given tuba pitch), look for enharmonic near-equivalents within a tolerance of $\pm 0.5 \phi$.
3.) If the row now has 2 or more possible horn pitches, indicate the maximum necessary tolerance and proceed to the next row.
4.) If the number of possible horn pitches is still 0 or 1 , proceed to the next tolerance range of $\pm 1.5 \phi$.
5.) Repeat this process until each row has at least 2 possible horn pitches.

In the composite table, some rows contain only ratios, some contain combinations of cents and ratios, and some contain only cents values. The intervals expressed as melodic distances are enharmonic near-equivalents of tuneable intervals within an indicated tolerance range. The resulting array may be 'navigated' by moving between rows and columns, producing a mathematical model of tuneable interval counterpoint between tuba and horn in which every tuba pitch and every horn pitch may be connected to other pitches by a tuneable path.

Because of the size of this array, only a small portion of it is reproduced in this paper. It exists as a searchable and sortable Excel spreadsheet, and may be obtained from the authors or on the website www.plainsound.org. The table has 6728 possible intervals in total.

318 rows have at least 1 exact ratio; 69 rows have intervals within a tolerance of $\pm 0.5 \phi ; 155$ rows have a tolerance of $\pm 1.5 \phi ; 17$ rows have a tolerance of $\pm 2.5 \phi ; 2$ rows have a tolerance of $\pm 3.5 \phi ; 2$ rows have a tolerance of $\pm 4.5 \phi$. Thus, 559 of 563 tuba pitches can be harmonized with horn pitches falling within $2.5 \phi$ of a tuneable interval, a highly acceptable tolerance for acoustic instruments. An audible version of this table has been programmed in MAX/MSP, producing available microtonal intervals between tuba and horn by clicking in a scrollable array.

Table 13


### 3.2 Notation of the available intervals

To investigate tuneable intervals with acoustic brass instruments, the various tables of pitches are written in staff notation. All 4782 exact tuneable ratios between tuba and horn, which were generated in the first array discussed above, have been converted to information compatible with Sibelius music notation software. The required Helmholtz-Ellis accidentals, the note names, the cents deviations, as well as MIDI data (note numbers and 14-bit pitch bend values) may be automatically calculated by transferring the pitch frequency ratios into a spreadsheet file ${ }^{14}$ conceived and programmed by the composer Stefan Bartling. The resulting text information is copied into his Sibelius plug-in "heHELIX", which translates the data into musical notation. The valve combination fingerings are added similarly, using his plug-ins "heHOFI" and "heTUFI". All of these tools are available from the authors.

Two examples of the results are given below - all possible occurrences of the intervals 13/10 and 10/13 (the inverted form simply implies that the horn is below the tuba), and all examples of the interval $11 / 1$ (note that this is one of the consonances greater than 3 octaves mentioned in section 1.4 above).

Table 14 (overleaf)

[^9]





### 3.3 Testing the tuneable intervals with tuba and horn

In order to test the full range between the tuba and horn, a book of music was prepared, notating theoretically tuneable intervals between tuba and horn. It also includes some wider tuneable intervals including $11 / 1$ (shown above) and 13/1. A systematic assessment of larger intervals is planned as part of our future work. To date, we have completed two working sessions with Dianna Gaetjens playing horn and Robin Hayward playing tuba. Here are a few general observations:
1.) The intervals empirically determined by earlier testing on string instruments are similarly tuneable on brass instruments.
2.) The theoretical analysis of available intervals yields combinations which may be accurately reproduced in practice. Unfamiliar valve combinations often required several attempts to be tuned accurately. In general, the players managed to play the intervals without having to substantially alter the pitches naturally occurring on their instruments.
3.) The very lowest tuba pedal tones are too indeterminate in pitch to produce tuneable sounds.
4.) The lowest tuba pitch producing good pitch resolution is $A 0$ (the lowest note of the piano).
5.) Intervals with melodic distance less than $1200 \phi$ are not tuneable in the very lowest register. In the bass register (the lower of the two pitches above F2) these intervals begin to demonstrate an audible periodic rumbling pattern which may be described as a tuneable noise. In the tenor register (the lower of the two pitches above C3) they begin to exhibit a clearly periodic composite sound, as was also found in tests with the same intervals on cello. In the highest register, the intervals produce very strongly audible combination tones reinforcing harmonics of their combined period. Once the horn pitches rise above D5, they become spectrally weaker and do not tune as easily with the tuba.
6.) In the low register, it is often easier to tune intervals when the horn is playing the lower pitch, because the spectral balance of its tone favors higher partials (it produces a reedy tone reminiscent of a bassoon). Conversely, the relatively strong fundamental of the tuba aids production of difference tones when it takes the lower voice, and these are most clearly evident in a higher register.
7.) As the intervals become melodically wider (> $2 / 1$ ), the lower limits of tuneability also descend in pitch.
8.) Many of the intervals wider than three octaves (> 8/1) are also tuneable and demand further systematic consideration.

Based on these tests it is possible to conclude that the theoretical analysis of available pitches produces results that are readily realizable by experienced musicians. These pitches can therefore serve as a basis for microtonal composition using tuneable intervals.

### 3.4 Some final remarks about working with tuneable intervals on brass instruments

Tuneable intervals offer one possible conceptual framework for composing in Just Intonation, allowing modulation between related harmonic and subharmonic series based on a set of perceptible sounds. In light of this question of perceptibility, our analysis of the brass tunings poses one difficulty: the pitches produced by the tuba's valve combinations introduce numerous fundamentals based on prime-number relationships higher than 23 (i.e., pitches which may not be reached by tuneable interval steps).

In any music which is based on audible harmonic steps, the logical consequences would be to either ignore these pitches as beings outside the limits of Fasslichkeit (Schoenberg's term for musical comprehensibility), to treat them as microtonal dissonances, or to consider contextual redefinitions of these pitches based on an acoustically acceptable enharmonic tolerance. Realizing subtle alterations of pitches is readily possible, since brass players are accustomed to making slight adjustments to naturally occurring pitches with their lips, by finetuning the slide position in the case of the trombone or by moving their hand in and out of the bell in the case of the horn.

Our current work includes developing tools for mapping limited subsets of 23-limit harmonic space, which may be searched for enharmonic near-equivalents to the brass pitches falling outside the perceptible harmonic space. In combination with the near-equivalents found in the tuba-horn tuneable interval chart, it is possible to conceptualize various means to harmonically employ the higher-prime fundamentals. It is hoped that these tools can serve as a provisional basis for compositional investigations of the brass instruments' microtonal possibilities and how they may be combined with other acoustic instruments in order to make music in Just Intonation.


[^0]:    ${ }^{1}$ The pitch notation used in this paper is based on MIDI note number $60=$ middle $\mathrm{C}=\mathrm{C} 4$.

[^1]:    ${ }^{2}$ Martin Vogel: "On the Relations of Tone" (Orpheus Verlag, Bonn, 1993), pp. 139-147

[^2]:    ${ }^{3}$ Harry Partch: "Genesis of a Music" (Second Edition, Da Capo Press, New York, 1974), p. 75.
    ${ }^{4}$ James Tenney: "John Cage and the Theory of Harmony" (1983), reprinted in "Soundings 13: The music of James Tenney" (Soundings Press, Santa Fe, 1984), p. 69.
    ${ }^{5}$ The musical term octave refers to the interval $2 / 1$ or $1200 \phi, n$ octaves is $2^{n} / 1$ or (1200n) $\phi$.
    ${ }^{6}$ Marc Sabat: "Analysis of Tuneable Intervals on Violin and Cello" (Plainsound Music Edition, Berlin, 2004)
    ${ }^{7}$ It was determined that the intervals $9 / 1,19 / 2,10 / 1,21 / 2,11 / 1,12 / 1,13 / 1,14 / 1$ and $15 / 1$ within the fourth octave may be easily tuned on horn and tuba.

[^3]:    ${ }^{8}$ Marc Sabat and Wolfgang von Schweinitz: "The Extended Helmholtz-Ellis JI Pitch Notation" (Plainsound Music Edition, Berlin, 2004)

[^4]:    ${ }^{9}$ Arnold Schoenberg: "Structural Functions of Harmony" (Second edition, revised, 1954, reprinted by Faber and Faber, London, 1983), pp. 19-20

[^5]:    ${ }^{10}$ The string length ratio $(\mathrm{k} / \mathrm{n})$ must be in lowest terms to produce the $\mathrm{n}^{\text {th }}$ partial at that node.
    ${ }^{11}$ Here it is assumed that all other parameters (tension, thickness, etc.) remain constant.

[^6]:    ${ }^{12}$ Martin Vogel: "On the Relations of Tone" (Orpheus Verlag, Bonn, 1993), pp. 373-385

[^7]:    ${ }^{13}$ In this section, there are two uses of the word "slide". The trigger valve-slide is set to a fixed interval and is used to alternate between the Bb and F sides of the tenorbass trombone. The slide refers to the continuously moveable tubing which allows for different fundamentals to be played on each side of the instrument.

[^8]:    * microtonal accidentals from The Extended Helmholtz-Ellis JI Pitch Notation, designed by Marc Sabat and Wolfgang von Schweinitz ** notated in F, sounding a 5th lower than written in both bass and treble clefs

[^9]:    14 "he_WRITE.xIs"

